Published by IOP Publishing for SISSA
Received: *August 15, 2008*
Accepted: *October 14, 2008*
Published: *November 5, 2008*

Electromagnetic effects in $K_{\ell 3}$ decays*

Vincenzo Cirigliano and Maurizio Giannotti

Theoretical Division, Los Alamos National Laboratory,
Los Alamos, NM 87545, U.S.A.
E-mail: cirigliano@lanl.gov, maurizio@lanl.gov

Helmut Neufeld

Fakultät für Physik, Universität Wien,
Boltzmanngasse 5, A-1090 Wien, Austria
E-mail: Helmut.Neufeld@univie.ac.at

Abstract: We study the radiative corrections to all $K_{\ell 3}$ decay modes to leading non-trivial order in the chiral effective field theory, working with a fully inclusive prescription on real photon emission. We present new results for $K_{\mu 3}$ modes and update previous results on K_{e3} modes. Our analysis provides important theoretical input for the extraction of the CKM element $V_{\rm us}$ from $K_{\ell 3}$ decays.

Keywords: Electromagnetic Processes and Properties, Chiral Lagrangians, Kaon Physics, Standard Model.

*This work was supported in part by the EU Contract No. MRTN-CT-2006-035482, "FLAVIAnet".

JHEP11(2008)006

Contents

1. Introduction

With the advent of precision measurements in Kaon physics (see [1] and references therein), $K_{\ell 3}$ decays offer the opportunity to probe charged current weak interactions at unprecedented levels. Most notably, with current experimental uncertainties, $K_{\ell 3}$ decays allow one to access the Cabibbo-Kobayashi-Maskawa (CKM) quark mixing angle $V_{\rm us}$ at the sub-percent level, and also provide competitive probes of lepton universality and the ratios of light quark masses. In order to fully exploit the amazing experimental achievements, it becomes mandatory to have theoretical control of these decays at the percent level or better. This requires quantitative understanding of the vector and scalar $K \to \pi$ form factors as well as the electromagnetic (EM) corrections. The framework to analyze the EM corrections is provided by Chiral Perturbation Theory (ChPT) [2], the low energy effective field theory (EFT) of QCD, extended to include the photon [3] and the light leptons [4] as active degrees of freedom. ChPT exploits the special role of π, K, η as Goldstone modes associated with the spontaneous breaking of chiral $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ symmetry, and provides a systematic expansion of the amplitudes in powers of the masses of pseudoscalar mesons and charged leptons ($p \sim M_{\pi,K,\ell}/\Lambda_\chi$ with $\Lambda_\chi \sim 4\pi F_\pi \sim 1.2\,\mathrm{GeV}$) and the electromagnetic coupling (e).

In this article we present results on the electromagnetic corrections to the four $K_{\ell 3}$ decay modes ($K = K^{\pm}, K^0; \ell = e, \mu$), based on a calculation of the amplitudes to leading non-trivial order in ChPT ($O(e^2 p^2)$). For all modes, we focus on (i) the electromagnetic

(EM) corrections to the Dalitz plot, which are needed to extract the momentum dependence of the $K \to \pi$ vector and scalar form factors from the experimental distribution; (ii) the integrated radiative correction to the decay rate, which is a crucial input in extracting the CKM mixing angle $V_{\rm us}$ from $K \to \pi \ell \nu [\gamma]$ decays. The theoretical framework for the calculation of electromagnetic contributions to $O(e^2 p^2)$ in $K_{\ell 3}$ decays was presented in ref. [5] and full numerical results on the K_{e3} modes were given in refs. [6], adopting a specific prescription for treating real photon emission and a specific factorization scheme for soft photons , which results in the *partial* inclusion of higher order terms in the chiral expansion. The novel features of the present work can be summarized as follows:

- Rather than using the soft-photon factorization procedure of ref. [5], we work here to *fixed chiral order* $e^2 p^2$, providing the complete corrections to decay distributions and total decay rates to $O(e^2 p^0)$.
- We give new results for $K_{\mu 3}$ modes and update our previous analysis of K_{e3} modes.
- We use a fully inclusive prescription for real photon emission, which is more appropriate for comparison with the experimental results.
- We update the structure-dependent EM correction, using the recent estimates of the relevant low-energy constants (LECs) provided in refs. [7, 8].

Preliminary results of our analysis have been made public in conference talks and proceedings [9, 10] and should be considered obsolete after the current publication. The paper is organized as follows: in section 2 we give an overview of various contributions to $K_{\ell 3}$ radiative corrections and derive the relevant master formula for the corrections to fixed chiral order ($O(e^2 p^0)$). In section 3 we present our results for differential and total radiative corrections, discussing their uncertainty. In section 4 we present our conclusions.

2. Radiative corrections to $K_{\ell 3}$ decays: overview

2.1 Generalities on $K_{\ell 3}$ decays

Let us briefly recall the main features of $K_{\ell 3}$ decays. The invariant amplitude for the process $K(p_K) \to \pi(p_\pi)\, \ell^+(p_\ell)\, \nu_\ell(p_\nu)$ reads

$$\mathcal{M} = \frac{G_{\rm F}}{\sqrt{2}} V_{\rm us}^* \; \bar{u}(p_\nu)\, \gamma^\mu \,(1-\gamma_5)\, v(p_\ell)\; C_K \left[f_+^{K\pi}(t)\,(p_K + p_\pi)_\mu + f_-^{K\pi}(t)\,(p_K - p_\pi)_\mu \right], \quad (2.1)$$

where $C_K = 1$ for $K^0_{\ell 3}$ and $C_K = 1/\sqrt{2}$ for $K^+_{\ell 3}$ modes. The expression in square brackets corresponds to the matrix element $\langle \pi(p_\pi) | V_\mu^{4-i5} | K(p_K) \rangle$, expressed in terms of the form factors $f_\pm^{K\pi}(t)$, which depend on the single variable $t = (p_K - p_\pi)^2$ and are known to $O(p^6)$ in ChPT [11–13]. To this order, a number of unknown LECs appear, of which only a few can be determined experimentally. A complete prediction to $O(p^6)$ requires theoretical input beyond ChPT, either from analytic approaches [14] or lattice QCD [15]. For phenomenological applications, it is common to parameterize the form factors $f_+(t)$

and $f_0(t) = f_+(t) + t/(M_K^2 - M_\pi^2) f_-(t)$ in terms of slope and curvature parameters[1] which can then be measured:

$$\bar{f}_+^{K\pi}(t) \equiv \frac{f_+^{K\pi}(t)}{f_+^{K\pi}(0)} = 1 + \lambda_+ \frac{t}{M_{\pi^\pm}^2} + \frac{1}{2}\lambda_+'' \frac{t^2}{M_{\pi^\pm}^4}, \tag{2.2}$$

$$\bar{f}_-^{K\pi}(t) \equiv \frac{f_-^{K\pi}(t)}{f_+^{K\pi}(0)} = \frac{M_K^2 - M_\pi^2}{M_{\pi^\pm}^2} \left(\lambda_0 - \lambda_+ - \frac{\lambda_+''}{2} \frac{t}{M_{\pi^\pm}^2} \right) . \tag{2.3}$$

The spin-averaged decay distribution depends on two independent kinematical variables, which we choose to be

$$z = \frac{2 p_\pi \cdot p_K}{M_K^2} = \frac{2E_\pi}{M_K}, \quad y = \frac{2 p_K \cdot p_\ell}{M_K^2} = \frac{2E_\ell}{M_K}, \tag{2.4}$$

where E_π (E_ℓ) is the pion (charged lepton) energy in the kaon rest frame, and M_K indicates the mass of the decaying kaon. Then the distribution (without radiative corrections) reads

$$\frac{d\Gamma^{(0)}}{dy\,dz} = \frac{G_{\rm F}^2\, |V_{\rm us}|^2\, M_K^5\, C_K^2}{128\,\pi^3}\, |f_+^{K\pi}(0)|^2\ \bar{\rho}^{(0)}(y,z), \tag{2.5}$$

$$\bar{\rho}^{(0)}(y,z) = A_1^{(0)}(y,z)\ |\bar{f}_+^{K\pi}(t)|^2 + A_2^{(0)}(y,z)\ \bar{f}_+^{K\pi}(t) \bar{f}_-^{K\pi}(t) + A_3^{(0)}(y,z)\ |\bar{f}_-^{K\pi}(t)|^2, \tag{2.6}$$

where the kinematical densities read ($r_\ell = (m_\ell/M_K)^2$ and $r_\pi = (m_\pi/M_K)^2$):

$$\begin{aligned} A_1^{(0)}(y,z) &= 4(z+y-1)(1-y) + r_\ell(4y+3z-3) - 4r_\pi + r_\ell(r_\pi - r_\ell), \\ A_2^{(0)}(y,z) &= 2r_\ell(3-2y-z+r_\ell-r_\pi), \\ A_3^{(0)}(y,z) &= r_\ell(1+r_\pi-z-r_\ell) . \end{aligned} \tag{2.7}$$

In the analysis of K_{e3} decays, the terms proportional to $A_{2,3}^{(0)}$ can be neglected, being proportional to $r_e \simeq 10^{-6}$. Finally, the decay rate reads

$$\Gamma^{(0)}(K_{\ell 3}) = \frac{G_{\rm F}^2\, |V_{\rm us}|^2\, M_K^5\, C_K^2}{128\,\pi^3}\, |f_+^{K\pi}(0)|^2\ I_{K\ell}^{(0)}(\lambda_i), \tag{2.8}$$

$$I_{K\ell}^{(0)}(\lambda_i) = \int_{\mathcal{D}_3} dy\,dz\ \bar{\rho}^{(0)}(y,z), \tag{2.9}$$

where the integral extends on the physical domain $\mathcal{D}_3$ defining the three-body Dalitz plot (see ref. [5] for the explicit definition).

2.2 Radiative corrections: soft factorization vs fixed chiral order

The above description of differential distributions and decay rates is modified by EM effects, which involve the emission of both virtual and real photons. Short distance electroweak corrections can be lumped in an overall factor $S_{\rm ew} = 1 + \frac{2\alpha}{\pi}\left(1 - \frac{\alpha_s}{4\pi}\right) \times \log\frac{M_Z}{M_\rho} + O(\frac{\alpha\alpha_s}{\pi^2})$, which is common to all semileptonic charged-current processes [17]. Long distance EM corrections to $K_{\ell 3}$ decays can be studied within ChPT. The leading non-trivial corrections

[1]See Ref [16] for a dispersive parameterization of the scalar form factor.

to the amplitudes appear to $O(e^2p^2)$ and imply corrections to the form factors, differential distributions, and decay rates starting to $O(e^2p^0)$.[2]

In ref. [5], it was argued that long distance EM effects can be taken into account by (i) a universal (i.e. non structure-dependent) shift in the densities $A_i^{(0)}(y,z)$ accompanied by (ii) structure-dependent corrections to the form factors $\bar{f}_{\pm}^{K\pi}(t)$. This result was obtained by factorizing out of the amplitude the universal soft photon corrections [18] that are sensitive only to charges, masses, and momenta of the particles involved in the decay. While this recipe has the benefit of being simple and elegant, it inherently mixes different orders in the chiral power counting (e.g. the soft-photon corrections proportional to $f_{-}^{K\pi}(t)$ only appear in the EFT calculation to $O(e^2p^4)$). As a consequence, the resulting decay distribution and rate contain not only the full chiral corrections of order e^2p^0 but also *incomplete* higher order corrections, generated by the factorization procedure. Since we are studying a fully photon-inclusive rate, there are no large logarithms associated with the factorized soft-photon corrections and therefore the partial higher order corrections that are included with the recipe of ref. [5] are *not* expected to give the dominant contributions to any given order: cancellations with unknown terms are possible. Motivated by this, in the present work we give the corrections to decay distributions and rates to fixed chiral order, namely $O(e^2p^0)$, to which the complete answer is known. We shall then use the comparison with the procedure of ref. [5] as a validation of our estimate of the theoretical uncertainty.

The virtual photon corrections to all $K_{\ell 3}$ amplitudes ($K = K^{\pm}, K^0$ and $\ell = e, \mu$) are known to $O(e^2p^2)$ [5, 6], while the real photon emission was worked out explicitly in those references only for K_{e3} modes (and only for a specific prescription on the treatment of real photon emission [19]). Our goal here is to provide a unified discussion of radiative corrections of $O(e^2p^0)$ to all $K_{\ell 3}$ decay rates, working with the fully inclusive prescription on real photon emission. We now sketch the derivation of the corrections induced by virtual and real photon emission to *fixed chiral order* $[O(e^2p^0)]$, and how they combine into a master formula for the inclusive rate.

2.2.1 Virtual photons

One-loop amplitudes involving virtual photons, together with the associated local counterterm contributions, induce an effective correction of $O(e^2p^0)$ to the QCD form factors $f_{\pm}^{K\pi}$, of the form:

$$f_{+}^{K\pi}(t) \to f_{+}^{K\pi}(t) + \delta f_{+}^{K\pi}(v) + \frac{\alpha}{4\pi}\,\Gamma_c(v, m_\ell^2, M_c^2; M_\gamma^2)\,, \tag{2.10}$$

$$f_{-}^{K\pi}(t) \to f_{-}^{K\pi}(t) + \delta f_{-}^{K\pi}(v)\,, \tag{2.11}$$

where M_c is the relevant charged meson mass and $v = u \equiv (p_K - p_\ell)^2$ for $K^{\pm}$ decays while $v = s \equiv (p_\pi + p_\ell)^2$ for K^0 decays. The function $\Gamma_c(v, m_\ell^2, M_c^2; M_\gamma^2)$ [5] encodes the universal soft photon virtual corrections and is infrared divergent (thus it depends explicitly on the infrared regulator M_γ). On the other hand, the corrections $\delta f_{\pm}^{K\pi}(v)$ encode structure

[2] Since $\bar{u}(p_\nu)\,\gamma^\mu\,(1-\gamma_5)\,v(p_\ell)\cdot(p_K \pm p_\pi)_\mu \sim O(p^2)$, EM corrections to $f_{\pm}^{K\pi}$ start at $O(e^2p^0)$. Moreover, since $y, z, r_\ell, r_\pi \sim O(1)$ we can book the densities $A_{1,2,3}^{(0)}$ as quantities of $O(1)$. Therefore, corrections of $O(e^2p^0)$ to $f_{\pm}^{K\pi}$ induce corrections to the decay distributions and rates of $O(e^2p^0)$ (see eqs. (2.6) and (2.8)).

dependent effects through one-loop corrections and chiral low-energy constants [5]. We report their expressions in appendix A in terms of functions defined in ref. [5].

Keeping in mind the chiral properties of $f_{\pm}^{K\pi}(t)$, namely that $f_+(t) = 1 + O(p^2)$ and $f_-(t) = O(p^2)$, the effect of virtual corrections to leading order in ChPT amounts to the following $O(e^2 p^0)$ shift to the differential distribution of eq. (2.6):

$$\delta\bar{\rho}^{\mathrm{EM-virtual}}(y,z) = A_1^{(0)}(y,z)\cdot\left[2\,\delta f_+^{K\pi}(v) + \frac{\alpha}{2\pi}\Gamma_c(v,m_\ell^2,M_c^2;M_\gamma^2)\right] + A_2^{(0)}(y,z)\cdot\delta f_-^{K\pi}(v) \quad . \tag{2.12}$$

2.2.2 Real photons

It is well known that only the inclusive sum of $K \to \pi\ell\nu$ and $K \to \pi\ell\nu + n\,\gamma$ (with $n = 1, 2, \ldots$) decay rates is infrared (IR) finite and observable. In the chiral power counting, the leading contribution to the radiative amplitudes $K(p_K) \to \pi(p_\pi)\,\ell^+(p_\ell)\,\nu_\ell(p_\nu)\gamma(k)$ is of $O(ep)$: this is all we need for the analysis of the rates to $O(e^2p^0)$. To this order the radiative amplitudes read:

$$\mathcal{M}_\gamma(K^+ \to \pi^0\ell^+\nu_\ell\gamma) = \frac{e\,G_{\mathrm{F}}}{\sqrt{2}} V_{\mathrm{us}}^* \, C_{K^+} \times \times\bar{u}(p_\nu)\left[(1+\gamma_5)(2\not{p}_\pi - m_\ell)\left(\frac{\epsilon^*\cdot p_\ell}{k\cdot p_\ell} - \frac{\epsilon^*\cdot p_K}{k\cdot p_K} + \frac{\not{k}\not{\epsilon}^*}{2k\cdot p_\ell}\right)\right]v(p_\ell)\,, \tag{2.13}$$

$$\mathcal{M}_\gamma(K^0 \to \pi^-\ell^+\nu_\ell\gamma) = \frac{e\,G_{\mathrm{F}}}{\sqrt{2}} V_{\mathrm{us}}^* \, C_{K^0} \times \times\bar{u}(p_\nu)\left[(1+\gamma_5)(2\not{p}_K + m_\ell)\left(\frac{\epsilon^*\cdot p_\ell}{k\cdot p_\ell} - \frac{\epsilon^*\cdot p_\pi}{k\cdot p_\pi} + \frac{\not{k}\not{\epsilon}^*}{2k\cdot p_\ell}\right)\right]v(p_\ell)\,. \tag{2.14}$$

The resulting correction to the differential (or total) decay rate can be calculated from

$$d\Gamma(K \to \pi\ell\nu\gamma) = \frac{1}{2M_K}\sum_{\mathrm{pol}} |\mathcal{M}_\gamma|^2 \, \frac{d\Omega_{\pi\ell\nu\gamma}}{(2\pi)^8}\,, \tag{2.15}$$

$$d\Omega_{\pi\ell\nu\gamma} = \prod_{i=\pi,\ell,\nu,\gamma} \frac{d^3p_i}{2\,p_i^0}\,\delta^{(4)}(p_K - p_\pi - p_\ell - p_\nu - k)\,, \tag{2.16}$$

where $d\Omega_{\pi\ell\nu\gamma}$ is the 4-body Lorentz invariant phase space.

The square modulus of the radiative amplitude can be decomposed into the sum of an IR singular term (T_{IR}) and an IR finite inner bremsstrahlung term (T_{IB}), as follows

$$\sum_{\mathrm{pol}} |\mathcal{M}_\gamma(K^0 \to \pi^-\ell^+\nu_\ell\gamma)|^2 = \frac{e^2G_F^2|V_{\mathrm{us}}|^2C_{K^0}^2}{2}\left[T_{\mathrm{IR}}^{K^0\ell} + T_{\mathrm{IB}}^{K^0\ell}\right], \tag{2.17}$$

$$\sum_{\mathrm{pol}} |\mathcal{M}_\gamma(K^+ \to \pi^0\ell^+\nu_\ell\gamma)|^2 = \frac{e^2G_F^2|V_{\mathrm{us}}|^2C_{K^+}^2}{2}\left[T_{\mathrm{IR}}^{K^+\ell} + T_{\mathrm{IB}}^{K^+\ell}\right], \tag{2.18}$$

with the IR singular pieces given by:

$$T_{\mathrm{IR}}^{K^0\ell} = 4\,M_K^4\,A_1^{(0)}(y,z) \times\left[-\frac{M_\pi^2}{(k\cdot p_\pi + \frac{M_\gamma^2}{2})^2} - \frac{m_\ell^2}{(k\cdot p_\ell + \frac{M_\gamma^2}{2})^2} + \frac{2p_\pi\cdot p_\ell}{(k\cdot p_\pi + \frac{M_\gamma^2}{2})(k\cdot p_\ell + \frac{M_\gamma^2}{2})}\right], \tag{2.19}$$

$$T_{\rm IR}^{K^+\ell} = 4\,M_K^4\,A_1^{(0)}(y,z) \times \left[-\frac{M_K^2}{(k\cdot p_K - \frac{M_\gamma^2}{2})^2} - \frac{m_\ell^2}{(k\cdot p_\ell + \frac{M_\gamma^2}{2})^2} + \frac{2 p_K\cdot p_\ell}{(k\cdot p_K - \frac{M_\gamma^2}{2})(k\cdot p_\ell + \frac{M_\gamma^2}{2})} \right] . \qquad (2.20)$$

The terms $T_{\rm IR}^{K\ell}$ generate an infrared divergence when integrated over the soft photon region of the four-body phase space, while the remaining terms are IR finite. Integrating over all variables except y and z, leads to a correction to the Dalitz plot density of the form

$$\delta\bar\rho^{\rm EM-real}(y,z) = A_1^{(0)}(y,z)\cdot\frac{\alpha}{\pi}\,I_0(y,z;M_\gamma) + \Delta_1^{\rm IB}(y,z)\,, \qquad (2.21)$$

where $I_0(y,z;M_\gamma)$ arises from $T_{\rm IR}$ while $\Delta_1^{\rm IB}$ from $T_{\rm IB}$. The functions $I_0(y,z;M_\gamma)$ and $\Delta_1^{\rm IB}(y,z)$ depend on the cut on hard real photon emission. Results based on integrating over all kinematically allowed photon energies are known analytically [5, 6, 19].

Finally, although both $I_0(y,z;M_\gamma)$ and $\Gamma_c(v,m_\ell^2,M_c^2;M_\gamma^2)$ are individually infrared divergent, they combine into the IR finite function

$$\Delta^{\rm IR}(y,z) = \frac{\alpha}{\pi}\left[I_0(y,z;M_\gamma) + \frac{1}{2}\Gamma_c(v,m_\ell^2,M^2;M_\gamma^2)\right]\,, \qquad (2.22)$$

and we end up with a finite correction to the Dalitz plot density:

$$\delta\bar\rho^{\rm EM}(y,z) = A_1^{(0)}(y,z)\cdot\left[\Delta^{\rm IR}(y,z) + 2\,\delta f_+^{K\pi}(v)\right] + \Delta_1^{\rm IB}(y,z) + A_2^{(0)}(y,z)\cdot\delta f_-^{K\pi}(v)\,. \qquad (2.23)$$

2.3 Master formula

After inclusion of both long-distance and short distance ($S_{\rm ew}$) radiative corrections, the differential decay distribution reads:

$$\frac{d\Gamma}{dy\,dz} = \frac{G_{\rm F}^2\,|V_{\rm us}|^2\,M_K^5\,C_K^2}{128\,\pi^3}\,S_{\rm ew}\,|f_+^{K\pi}(0)|^2\left[\bar\rho^{(0)}(y,z) + \delta\bar\rho^{\rm EM}(y,z)\right]. \qquad (2.24)$$

This expression should be the basis to properly determine experimentally the momentum-dependence of the QCD form factors $\bar f_\pm(t)$ appearing in $\bar\rho^{(0)}(y,z)$.

Corrections to the decay rate are obtained by integrating over the variables y and z. In the case of fully inclusive prescription on the radiated photon, the real photon EM correction should be integrated not only over the 3-body region $\mathcal{D}_3$ but on the whole region $\mathcal{D}_4$ allowed by 4-body kinematics. Taking into account all these corrections, the master formula for $K_{\ell 3}$ decay rates reads:

$$\Gamma(K_{\ell 3[\gamma]}) = \frac{G_{\rm F}^2\,|V_{\rm us}|^2\,M_K^5\,C_K^2}{128\,\pi^3}\,S_{\rm ew}\,|f_+^{K^0\pi^-}(0)|^2\,I_{K\ell}^{(0)}(\lambda_i)\left[1+\delta_{\rm EM}^{K\ell}+\delta_{\rm SU(2)}^{K\pi}\right], \qquad (2.25)$$

where the strong isospin breaking correction is

$$\delta_{\rm SU(2)}^{K\pi} \equiv \left(\frac{f_+^{K\pi}(0)}{f_+^{K^0\pi^-}(0)}\right)^2 - 1 \qquad (2.26)$$

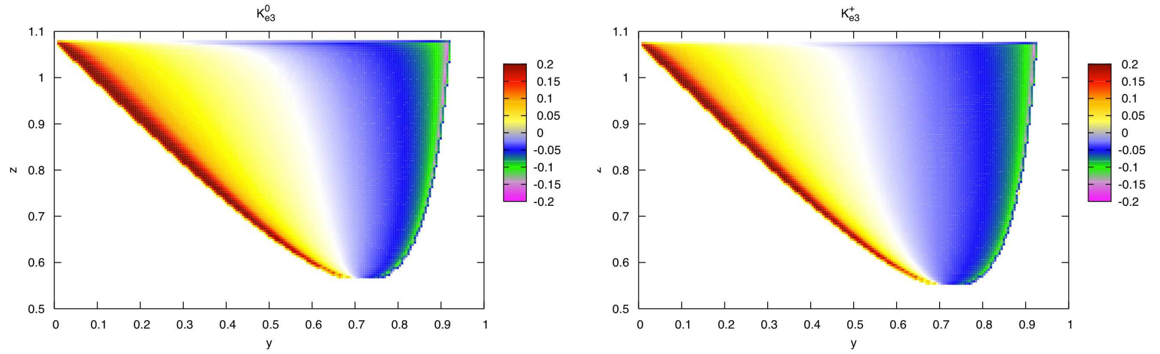

Figure 1: Density plot of the EM correction to the differential distribution $(\delta\bar{\rho}^{\rm EM}(y,z)/\bar{\rho}^{(0)}(\lambda_i)(y,z))$ of K^0_{e3} (left panel) and $K^{\pm}_{e3}$ (right panel).

and the EM radiative correction

$$\delta^{K\ell}_{\rm EM} = \delta^{K\ell}_{\rm EM}(\mathcal{D}_3) + \delta^{K\ell}_{\rm EM}(\mathcal{D}_{4-3}) \tag{2.27}$$

receives contributions from the 3-body and 4-body kinematical regions:

$$\delta^{K\ell}_{\rm EM}(\mathcal{D}_3) = \frac{1}{I^{(0)}_{K\ell}(\lambda_i)} \cdot \int_{\mathcal{D}_3} dy\, dz\ \delta\bar{\rho}^{\rm EM}(y,z)\,, \tag{2.28}$$

$$\delta^{K\ell}_{\rm EM}(\mathcal{D}_{4-3}) = \frac{1}{I^{(0)}_{K\ell}(\lambda_i)} \cdot \frac{\alpha}{2\,\pi^4 M_K^6} \int_{\mathcal{D}_{4-3}} d\Omega_{\pi\ell\nu\gamma}\ \left(T^{K\ell}_{\rm IR} + T^{K\ell}_{\rm IB}\right)\ . \tag{2.29}$$

Note that Ginsberg's prescription [19] for real photon emission, which was adopted in refs. [5, 6], amounts to discarding the integral in eq. (2.29).

3. Results and discussion

The main outcome of our analysis are the differential corrections $\delta\bar{\rho}^{\rm EM}(y,z)$ to the Dalitz plot density (eq. (2.23)) and the integrated corrections $\delta^{K\ell}_{\rm EM}(\mathcal{D}_3)$ and $\delta^{K\ell}_{\rm EM}(\mathcal{D}_{4-3})$ (eqs. (2.28) and (2.29) respectively). $\delta\bar{\rho}^{\rm EM}(y,z)$ is known analytically through the work of refs. [5, 6, 19]. The integration needed to calculate $\delta^{K\ell}_{\rm EM}(\mathcal{D}_3)$ has been performed with the Gauss quadrature method. On the other hand, the integration needed to calculate $\delta^{K\ell}_{\rm EM}(\mathcal{D}_{4-3})$ has been performed with a Monte Carlo technique based on the RAMBOS event generator [20].

In order to give numerical results, we have to specify the input parameters. The differential and integrated EM corrections to $O(e^2p^0)$ depend on a number of LECs of ChPT. The EM LECs are given by convolutions of appropriate QCD correlators with known kernels. They have been recently estimated in refs. [7, 8] by replacing the QCD correlators with meromorphic approximants, in the spirit of the large-N_C expansion. We use the results of refs. [7, 8] for our central values, and conservatively assign 100% fractional uncertainty to the LECs.

The integrated corrections $\delta^{K\ell}_{\rm EM}(\mathcal{D}_3)$ and $\delta^{K\ell}_{\rm EM}(\mathcal{D}_{4-3})$ also depend, through the normalization factor $I^{(0)}_{K\ell}(\lambda_i)$, on the slope and curvature of the scalar and vector form factors, namely λ_+, λ''_+, and λ_0. For these quantities we use the results of a global fit to all consistent experimental data (i.e. without including the NA48 result) as reported in ref. [1]:

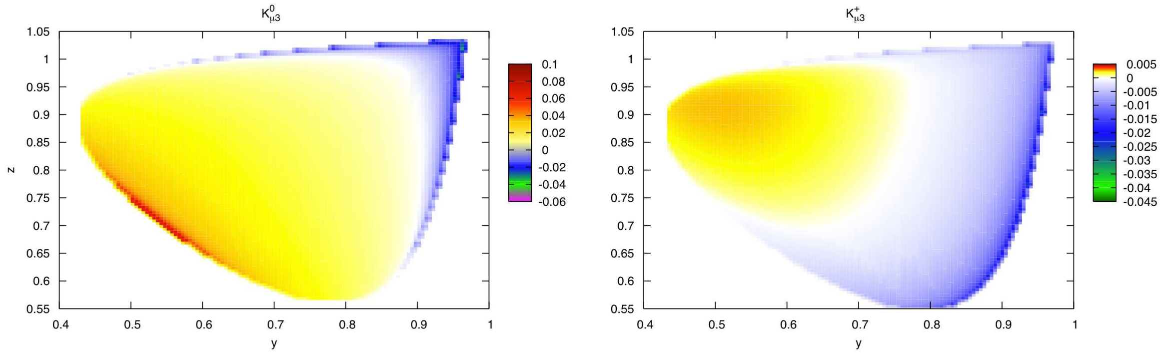

Figure 2: Density plot of the EM correction to the differential distribution $(\delta\bar{\rho}^{\mathrm{EM}}(y,z)/\bar{\rho}^{(0)}(\lambda_i)(y,z))$ of $K^0_{\mu 3}$ (left panel) and $K^{\pm}_{\mu 3}$ (right panel).

$\lambda_+ = (25.0 \pm 0.8) \cdot 10^{-3}$, $\lambda''_+ = (1.6 \pm 0.4) \cdot 10^{-3}$, $\lambda_0 = (16.0 \pm 0.8) \cdot 10^{-3}$. The choice of this reference set of input parameters is very simple but somewhat inconsistent as the charged and neutral K parameters are distinguished by strong isospin breaking and EM effects [13, 23]. However, one should keep in mind that the uncertainty on $I^{(0)}_{K\ell}(\lambda_i)$ induced by the λ_i is below the percent level and is completely negligible in the analysis of $\delta^{K\ell}_{\mathrm{EM}}$ (although it has some impact on the extraction of V_{us} through eq. (2.25)). In table 1 we provide the $I^{(0)}_{K\ell}(\lambda_i)$ corresponding to our choice λ_i, so the reader can easily convert our results for $\delta^{K\ell}_{\mathrm{EM}}$ to any choice of slope parameters with a simple re-scaling.

3.1 Corrections to the Dalitz plot

In figures 1 and 2 we show a density plot of the ratio $\delta\bar{\rho}^{\mathrm{EM}}/\bar{\rho}^{(0)}(\lambda_i)$ for K^0_{e3}, $K^{\pm}_{e3}$, $K^0_{\mu 3}$ and $K^{\pm}_{\mu 3}$ as a function of the variables y, z, corresponding to the input on λ_i and LECs specified in the previous subsection.[3] The theoretical uncertainty on the LECs and higher order corrections induces an overall uncertainty of about $\pm 0.3\%$ in $\delta\bar{\rho}^{\mathrm{EM}}/\bar{\rho}^{(0)}(\lambda_i)$. It is important to notice that the correction to the Dalitz distribution can be locally large ($O(10\%)$) and does not have definite sign, implying possible cancellations in the integrated total EM correction.

3.2 Corrections to the decay rates

Table 1 summarizes the numerical results of the long-distance radiative corrections to fixed order e^2p^0, obtained using the central values for the LECs, slopes, and curvature of the form factors as described above.

Two prominent features of the results in table 1 can be understood on a qualitative level. First, the EM corrections to the neutral K decays are expected to be positive and sizable on account of the Coulomb final state interaction term between ℓ^+ and π^-, that produces a correction factor of $\pi\alpha/v^{\mathrm{rel}}_{\ell^+\pi^-} \sim 2\%$ over most of the Dalitz plot. While the exact correction and the relative size of $K^0_{\mu 3}$ and K^0_{e3} depend on other effects such as the emission of real photons, the qualitative expectation based on Coulomb interaction is confirmed by the detailed calculation. Second, the large hierarchy $\delta^{K\mu}_{\mathrm{EM}}(\mathcal{D}_{4-3}) \ll \delta^{Ke}_{\mathrm{EM}}(\mathcal{D}_{4-3})$ admits a simple interpretation in terms of *bremsstrahlung* off the charged lepton in the

[3] Numerical tables for these corrections are available from the authors upon request.

final state. The probability of emitting soft photons is a function of the lepton velocity v_ℓ which becomes logarithmically singular as $v_\ell \to 1$, thus enhancing the electron emission. For typical values of v_ℓ in $\mathcal{D}_{4-3}$, the semiclassical emission probability [22] implies $\delta_{\rm EM}^{Ke}(\mathcal{D}_{4-3})/\delta_{\rm EM}^{K\mu}(\mathcal{D}_{4-3}) \sim 20 \to 40$.

The theoretical uncertainty to be assigned to $\delta_{\rm EM}^{K\ell}$ arises from two sources: the input parameters (LECs, λ_i) appearing in the $O(e^2p^0)$ correction and unknown higher order terms in the EFT expansion, starting at $O(e^2p^2)$. For the parametric uncertainty we find that:

- Experimental errors on the form factor parameters λ_+, $\lambda_+^{''}$, and λ_0 induce a fractional uncertainty in $I_{K\ell}^{(0)}(\lambda_i)$ and in $\delta_{\rm EM}^{K\ell}$ well below the percent level [1]. We can safely ignore this source of uncertainty in $\delta_{\rm EM}^{K\ell}$.

- To the order we work, the electromagnetic LECs contribute a v-independent term to $\delta f_\pm^{K\pi}(v)$, which thus affects the decay rates as follows,

$$\delta_{\rm EM}^{K\ell}(\mathcal{D}_3) \ \sim \ 2\,\delta f_+\Big|_{\rm LECs} \cdot \frac{\int_{\mathcal{D}_3} dydz A_1^{(0)}(y,z)}{I_{K\ell}^{(0)}(\lambda_i)} + \delta f_-\Big|_{\rm LECs} \cdot \frac{\int_{\mathcal{D}_3} dydz A_2^{(0)}(y,z)}{I_{K\ell}^{(0)}(\lambda_i)} \ . \qquad (3.1)$$

  The coefficient of $2\,\delta f_+$ is $O(1)$ while the coefficient of δf_- is roughly 0.2 for $K_{\mu 3}$ decays and completely negligible for K_{e3} decays, being $O(m_e/M_K)^2$. Taking a very conservative attitude, we assign a 100% fractional uncertainty to the LECs X_1 and $X_6^{\rm phys}$ contributing to δf_+ (we use LECs central values from refs. [7, 8]). Treating the LEC errors as independent leads to an absolute uncertainty of $\pm 0.11\%$ in $\delta_{\rm EM}^{K^0\ell}$ and of $\pm 0.16\%$ in $\delta_{\rm EM}^{K^\pm\ell}$.

In order to discuss the error coming from higher order chiral corrections not included in our analysis, we find it useful to decompose (to each chiral order) the EM corrections $\delta^{K\ell}$ in terms of $\delta_{1,2,3,4}$:

$$\begin{aligned} \delta^{K^0 e} &= \delta_1 + \delta_2 + \delta_3 + \delta_4 \,, \\ \delta^{K^0 \mu} &= \delta_1 + \delta_2 - \delta_3 - \delta_4 \,, \\ \delta^{K^\pm e} &= \delta_1 - \delta_2 + \delta_3 - \delta_4 \,, \\ \delta^{K^\pm \mu} &= \delta_1 - \delta_2 - \delta_3 + \delta_4 \ . \end{aligned} \qquad (3.2)$$

Here δ_1 represents a correction common to all modes, δ_2 a correction anti-correlated in kaon isospin but blind to lepton flavor, and finally $\delta_{3,4}$ are lepton-universality breaking terms, correlated and anti-correlated in kaon isospin, respectively. To $O(e^2p^0)$ we find $\delta_1^{e^2p^0} = 0.63\%$, $\delta_2^{e^2p^0} = 0.57\%$, $\delta_3^{e^2p^0} = -0.08\%$, $\delta_4^{e^2p^0} = -0.12\%$. On the basis of chiral power counting we expect the higher order corrections to scale as

$$\delta_i^{e^2p^2} \sim (M_K/(4\pi F_\pi))^2 \cdot \delta_i^{e^2p^0} \sim 0.2 \cdot \delta_i^{e^2p^0} \ . \qquad (3.3)$$

This estimate is validated by comparison of the fixed chiral order results with the ones obtained within the "soft-photon factorization" approach discussed in section 4 of ref. [5],

	$I^{(0)}_{K\ell}(\lambda_i)$	$\delta^{K\ell}_{\rm EM}(\mathcal{D}_3)(\%)$	$\delta^{K\ell}_{\rm EM}(\mathcal{D}_{4-3})(\%)$	$\delta^{K\ell}_{\rm EM}(\%)$
K^0_{e3}	0.103070	0.50	0.49	0.99 ± 0.22
$K^\pm_{e3}$	0.105972	-0.35	0.45	0.10 ± 0.25
$K^0_{\mu3}$	0.068467	1.38	0.02	1.40 ± 0.22
$K^\pm_{\mu3}$	0.070324	0.007	0.009	0.016 ± 0.25

Table 1: Summary of phase space integrals and EM corrections to the $K_{\ell3}$ decay rates. The EM corrections are calculated to fixed order in ChPT ($O(e^2p^0)$). The phase space integrals are calculated using slope and curvature parameters from the fit of ref. [1]. The uncertainty estimate is discussed in the text.

which include a class of higher order chiral corrections (see table 2).[4] The only anomaly appears to be in the coefficient δ_3, where we find $\delta_3 :\ -0.08\% \to -0.16\%$ when going from fixed chiral order to the soft factorization scheme. This can be traced back to the cancellation between the negative contribution from $\mathcal{D}_3$ (-0.31%) and the positive contribution from $\mathcal{D}_{4-3}$ (0.23%). Multiplying these individual pieces by 0.2 gives $\sim$ 0.06 and $\sim$ 0.05, respectively, which is just the order of magnitude of the shift we are seeing (-0.08 → -0.16).

Based on the above discussion, we bound the higher order uncertainties as follows: $|\delta_1^{e^2p^2}| < 0.13\%$, $|\delta_2^{e^2p^2}| < 0.11\%$, $|\delta_3^{e^2p^2}| < 0.08\%$, $|\delta_4^{e^2p^2}| < 0.025\%$. Assuming that the uncertainties from δ_i and from each LEC are uncorrelated, we arrive to the errors quoted in table 1 for the total corrections, with correlation matrix given by:

$$\begin{pmatrix} 1 & +0.081 & +0.685 & -0.147 \\ & 1 & -0.147 & +0.764 \\ & & 1 & +0.081 \\ & & & 1 \end{pmatrix} . \tag{3.4}$$

Finally, the above results allow us to evaluate the uncertainties on the linear combinations of $\delta^{K\ell}_{\rm EM}$ which are relevant for lepton universality and strong isospin-breaking tests:

$$\delta^{K^0e}_{\rm EM} - \delta^{K^0\mu}_{\rm EM} = -(0.41 \pm 0.17)\% , \tag{3.5}$$
$$\delta^{K^\pm e}_{\rm EM} - \delta^{K^\pm\mu}_{\rm EM} = (0.08 \pm 0.17)\% , \tag{3.6}$$
$$\delta^{K^\pm e}_{\rm EM} - \delta^{K^0 e}_{\rm EM} = -(0.89 \pm 0.32)\% , \tag{3.7}$$
$$\delta^{K^\pm\mu}_{\rm EM} - \delta^{K^0\mu}_{\rm EM} = -(1.38 \pm 0.32)\% . \tag{3.8}$$

4. Conclusions

In this work we have provided a unified discussion of the radiative corrections of $O(e^2p^0)$ to all $K_{\ell3}$ decay rates. We have argued that through the calculation of $K \to \pi\ell\nu$ amplitudes

[4]Note that in section 5.3 of ref. [5] an alternative factorization prescription is given, which is valid only for K_{e3} modes. The latter prescription was used in the numerical analysis of refs. [5, 6] and would lead to results slightly different from those of table 2. The first two entries in the first column should be replaced as follows: 0.41 → 0.56 and −0.564 → −0.41.

	$\delta^{K\ell}_{\rm EM}(\mathcal{D}_3)(\%)$	$\delta^{K\ell}_{\rm EM}(\mathcal{D}_{4-3})(\%)$	$\delta^{K\ell}_{\rm EM}(\%)$
K^0_{e3}	0.41	0.59	1.0
$K^{\pm}_{e3}$	-0.564	0.528	-0.04
$K^0_{\mu 3}$	1.57	0.04	1.61
$K^{\pm}_{\mu 3}$	-0.006	0.011	0.005

Table 2: Summary of EM corrections to the $K_{\ell 3}$ decay rates calculated according to the "soft-photon factorization" approach of ref. [5], which includes *incomplete* higher order terms in the chiral expansion. Comparison with the results of table 1 validates our estimate of the theoretical uncertainties.

to $O(e^2p^2)$ in the chiral effective theory we can derive the complete corrections of $O(e^2p^0)$ to the Dalitz plot density and the integrated decay rate. We have systematically discarded higher order effects that are only partially known, and we have included the unknown effects in a generous estimate of the theoretical uncertainty. For the first time we have presented complete numerical results for the $K_{\mu 3}$ modes, while also updating the previous analysis of K_{e3} modes.

The main outcome of our investigation is summarized in table 1, which contains the corrections to the total (fully photon inclusive) decay rates of all $K_{\ell 3}$ decay modes ($K = K^{\pm}, K^0; \ell = e, \mu$). These results provide important theoretical input for the determination of the product $f_+^{K^0\pi^-}(0) \cdot V_{\rm us}$ from $K_{\ell 3}$ decays at the 0.2% level [1] through eq. (2.25). This task requires as additional theoretical input the factor $\delta^{K\pi}_{\rm SU(2)}$, which has been recently updated in refs. [13, 23]. We refrain here from producing a number for $f_+^{K^0\pi^-}(0) \cdot V_{\rm us}$: the result reflecting most recent experimental data can be found in the Flavianet Kaon Working Group web page [24].

Further reduction of the theoretical uncertainty on the corrections $\delta^{\rm EM}_{K\ell}$ would require an analysis of the amplitudes to $O(e^2p^4)$ in ChPT, which is beyond the scope of this work. At the moment there appears to be no immediate need for such an analysis, since the error on $V_{\rm us}$ is dominated by the $\sim 1\%$ theoretical uncertainty in $f_+^{K^0\pi^-}(0)$, for which a compilation and discussion of theoretical results can be found in ref. [1].

Acknowledgments

The work of V.C. is supported by the U.S. DOE Office of Science and by the LDRD Program at Los Alamos National Laboratory. H.N. thanks Julia Schweizer for collaboration at an early stage of this work. V.C. thanks Gino Isidori for useful discssions. We thank Alex Friedland for input concerning the figures and Matthew Moulson for input concerning the error analysis.

A. Corrections induced by virtual photons: $\delta f_{\pm}^{K\pi}(v)$

Using the notation of refs. [5, 6] with $f_{\pm}^{\mathrm{EM-loc}} \equiv \widehat{f}_{\pm}$, we have:

$$\delta f_{+}^{K^+\pi^0}(u) = \frac{\alpha}{4\pi}\Big[\Gamma_1(u,m_\ell^2,M_K^2)+\Gamma_2(u,m_\ell^2,M_K^2)\Big] + \widehat{f}_{+}^{\mathrm{K}^+\pi^0}\,, \tag{A.1}$$

$$\delta f_{-}^{K^+\pi^0}(u) = \frac{\alpha}{4\pi}\Big[\Gamma_1(u,m_\ell^2,M_K^2)-\Gamma_2(u,m_\ell^2,M_K^2)\Big] + \widehat{f}_{-}^{\mathrm{K}^+\pi^0}\,, \tag{A.2}$$

and

$$\delta f_{+}^{K^0\pi^-}(s) = \frac{\alpha}{4\pi}\Big[\Gamma_1(s,m_\ell^2,M_\pi^2)+\Gamma_2(s,m_\ell^2,M_\pi^2)\Big] + \widehat{f}_{+}^{\mathrm{K}^0\pi^-}\,, \tag{A.3}$$

$$\delta f_{-}^{K^0\pi^-}(s) = \frac{\alpha}{4\pi}\Big[\Gamma_2(s,m_\ell^2,M_\pi^2)-\Gamma_1(s,m_\ell^2,M_\pi^2)\Big] + \widehat{f}_{-}^{\mathrm{K}^0\pi^-}\,. \tag{A.4}$$

References

[1] FlaviaNet Working Group on Kaon Decays collaboration, M. Antonelli et al., *Precision tests of the standard model with leptonic and semileptonic kaon decays*, arXiv:0801.1817.

[2] S. Weinberg, *Phenomenological Lagrangians*, *Physica* **A 96** (1979) 327;
J. Gasser and H. Leutwyler, *Chiral perturbation theory to one loop*, *Ann. Phys. (NY)* **158** (1984) 142;
H. Leutwyler, *On the foundations of chiral perturbation theory*, *Ann. Phys. (NY)* **235** (1994) 165 [hep-ph/9311274].

[3] R. Urech, *Virtual photons in chiral perturbation theory*, *Nucl. Phys.* **B 433** (1995) 234 [hep-ph/9405341];
H. Neufeld and H. Rupertsberger, *Isospin breaking in chiral perturbation theory and the decays $\eta \to \pi\ell\nu$ and $\tau \to \eta\pi\nu$*, *Z. Physik* **C 68** (1995) 91; *The electromagnetic interaction in chiral perturbation theory*, *Z. Physik* **C 71** (1996) 131 [hep-ph/9506448].

[4] M. Knecht, H. Neufeld, H. Rupertsberger and P. Talavera, *Chiral perturbation theory with virtual photons and leptons*, *Eur. Phys. J.* **C 12** (2000) 469 [hep-ph/9909284].

[5] V. Cirigliano, M. Knecht, H. Neufeld, H. Rupertsberger and P. Talavera, *Radiative corrections to $K_{\ell 3}$ decays*, *Eur. Phys. J.* **C 23** (2002) 121 [hep-ph/0110153].

[6] V. Cirigliano, H. Neufeld and H. Pichl, *K_{e3} decays and CKM unitarity*, *Eur. Phys. J.* **C 35** (2004) 53 [hep-ph/0401173].

[7] S. Descotes-Genon and B. Moussallam, *Radiative corrections in weak semi-leptonic processes at low energy: a two-step matching determination*, *Eur. Phys. J.* **C 42** (2005) 403 [hep-ph/0505077].

[8] B. Ananthanarayan and B. Moussallam, *Four-point correlator constraints on electromagnetic chiral parameters and resonance effective Lagrangians*, *JHEP* **06** (2004) 047 [hep-ph/0405206].

[9] H. Neufeld, *Electromagnetic contributions to $K_{\ell 3}$ decays*, talk presented at *Mini-Workshop on K physics*, INFN Frascati Italy May 18–19 2007, available at http://www.lnf.infn.it.

[10] V. Cirigliano, *Precision tests of the standard model with $K_{\ell 3}$ decays*, PoS(KAON)007.

[11] P. Post and K. Schilcher, *$K_{\ell 3}$ form factors at order p^6 in chiral perturbation theory*, *Eur. Phys. J.* **C 25** (2002) 427 [hep-ph/0112352].

[12] J. Bijnens and P. Talavera, *$K_{\ell 3}$ decays in chiral perturbation theory*, *Nucl. Phys.* **B 669** (2003) 341 [hep-ph/0303103], see also http://www.thep.lu.se/~bijnens/chpt.html.

[13] J. Bijnens and K. Ghorbani, *Isospin breaking in $K\pi$ vector form-factors for the weak and rare decays $K_{\ell 3}$, $K \to \pi\nu\bar{\nu}$ and $K \to \pi\ell^+\ell^-$*, arXiv:0711.0148.

[14] H. Leutwyler and M. Roos, *Determination of the elements $V_{\rm us}$ and $V_{\rm ud}$ of the Kobayashi-Maskawa matrix*, *Z. Physik* **C 25** (1984) 91;
M. Jamin, J.A. Oller and A. Pich, *Order p^6 chiral couplings from the scalar $K\pi$ form factor*, *JHEP* **02** (2004) 047 [hep-ph/0401080];
V. Cirigliano et al., *The $\langle SPP \rangle$ Green function and* SU(3) *breaking in $K_{\ell 3}$ decays*, *JHEP* **04** (2005) 006 [hep-ph/0503108].

[15] D. Bećirević et al., *The $K \to \pi$ vector form factor at zero momentum transfer on the lattice*, *Nucl. Phys.* **B 705** (2005) 339 [hep-ph/0403217];
Fermilab Lattice collaboration, M. Okamoto, *Full CKM matrix with lattice QCD*, hep-lat/0412044;
JLQCD collaboration, N. Tsutsui et al., *Kaon semileptonic decay form factors in two-flavor QCD*, PoS(LAT2005)357 [hep-lat/0510068];
C. Dawson, T. Izubuchi, T. Kaneko, S. Sasaki and A. Soni, *Vector form factor in $K_{\ell 3}$ semileptonic decay with two flavors of dynamical domain-wall quarks*, *Phys. Rev.* **D 74** (2006) 114502 [hep-ph/0607162];
D.J. Antonio et al., *$K \to \pi\ell\nu$ form factor with $N_f = 2 + 1$ dynamical domain wall fermions*, PoS(LAT2006)101 [hep-lat/0610080];
P.A. Boyle et al., *$K_{\ell 3}$ semileptonic form factor from $2 + 1$ flavour lattice QCD*, *Phys. Rev. Lett.* **100** (2008) 141601 [arXiv:0710.5136].

[16] V. Bernard, M. Oertel, E. Passemar and J. Stern, *$K^L_{\mu 3}$ decay: a stringent test of right-handed quark currents*, *Phys. Lett.* **B 638** (2006) 480 [hep-ph/0603202].

[17] A. Sirlin, *Current algebra formulation of radiative corrections in gauge theories and the universality of the weak interactions*, *Rev. Mod. Phys.* **50** (1978) 573 [*Erratum ibid.* **50** (1978) 905]; *Large m_W, m_Z behavior of the $O(\alpha)$ corrections to semileptonic processes mediated by W*, *Nucl. Phys.* **B 196** (1982) 83;
W.J. Marciano and A. Sirlin, *Radiative corrections to $\pi_{\ell 2}$ decays*, *Phys. Rev. Lett.* **71** (1993) 3629.

[18] D.R. Yennie, S.C. Frautschi and H. Suura, *The infrared divergence phenomena and high-energy processes*, *Ann. Phys. (NY)* **13** (1961) 379;
S. Weinberg, *Infrared photons and gravitons*, *Phys. Rev.* **140** (1965) B516.

[19] E.S. Ginsberg, *Radiative corrections to K^0_{e3} decays and the $\Delta I = 1/2$ rule*, *Phys. Rev.* **171** (1968) 1675 [*Erratum ibid.* **174** (1968) 2169]; *Radiative corrections to the $K^{\pm}_{e3}$ Dalitz plot*, *Phys. Rev.* **162** (1967) 1570 [*Erratum ibid.* **187** (1969) 2280]; *Radiative corrections to $K_{\mu 3}$ decays*, *Phys. Rev.* **D 1** (1970) 229.

[20] R. Kleiss, W.J. Stirling and S.D. Ellis, *A new Monte Carlo treatment of multiparticle phase space at high-energies*, *Comput. Phys. Commun.* **40** (1986) 359.

[21] Particle Data Group collaboration, W.M. Yao et al., *Review of particle physics*, *J. Phys.* **G 33** (2006) 1.

[22] J.D. Jackson, *Classical electrodynamics*, 3rd edition, Wiley, New York U.S.A. (1998).

[23] A. Kastner and H. Neufeld, *The $K_{\ell 3}$ scalar form factors in the standard model*, arXiv:0805.2222.

[24] The Flavianet Kaon Working Group maintains updated information at the following URL http://www.lnf.infn.it/wg/vus.